XVI. *On the Arithmetic of Impoſſible Quantities. By the Rev.* John Playfair, *A. M. Communicated by the Rev.* Nevil Maſkelyne, *D. D. F. R. S. and Aſtronomer Royal.*

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THE paradoxes which have been introduced into algebra, and remain unknown in geometry, point out a very remarkable difference in the nature of thoſe ſciences. The propoſitions of geometry have never given riſe to controverſy, nor needed the ſupport of metaphyſical diſcuſſion. In algebra, on the other hand, the doctrine of negative quantities and its conſequences have often perplexed the analyſt, and involved him in the moſt intricate diſputations. The cauſe of this diverſity, in ſciences which have the ſame object, muſt no doubt be ſought for in the different modes which they employ to expreſs our ideas. In geometry every magnitude is repreſented by one of the ſame kind; lines are repreſented by a line, and angles by an angle. The genus is always ſignified by the individual, and a general idea by one of the particulars which fall under it. By this means all contradiction is avoided, and

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the geometer is never permitted to reaſon about the relations of things which do not exiſt, or cannot be exhibited. In algebra again every magnitude being denoted by an artificial ſymbol, to which it has no reſemblance, is liable, on ſome occaſions, to be neglected, while the ſymbol may become the ſole object of attention. It is not perhaps obſerved where the cennection between them ceaſes to exiſt, and the analyſt continues to reaſon about the characters after nothing is left which they can poſſibly expreſs: if then, in the end, the concluſions which hold only of the characters be transferred to the quantities themſelves, obſcurity and paradox muſt of neceſſity enſue. The truth of theſe obſervations will be rendered evident by conſidering the nature of imaginary expreſſions, and the different uſes to which they have been applied.

2. Thoſe expreſſions, as is well known, owe their origin to a contradiction taking place in that combination of ideas which they were intended to denote. Thus, if it be required to divide the given line AB (fig. 1.) $= a$ in C, ſo that AC × CB may be equal to a given ſpace b^2, and if $\text{AC}=x$, then $x=\frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2-b^2}$; which value of x is imaginary when b^2 is greater than $\frac{1}{4}a^2$; now to ſuppoſe that b^2 is greater than $\frac{1}{4}a^2$, is to ſuppoſe that the rectangle AC × CB is greater than the ſquare of half the line AB,

which is impoſſible. The ſame holds wherever expreſſions of this kind occur. Thus, when it is aſſerted that unity has the three cube roots 1, $\frac{-1+\sqrt{-3}}{2}$, $\frac{-1-\sqrt{-3}}{2}$, no more is meant than that when the general equation $x^3-ax^2+bx-r=0$ is, by a change in the data, reduced to the particular ſtate $x^3-1=0$, x is then equal to unity only, and admits not of any other value, as it does in more general forms of the equation. The natural office of imaginary expreſſions is, therefore, to point out when the conditions, from which a general formula is derived, become inconſiſtent with each other; and they correſpond in the algebraic calculus to that part of the geometrical analyſis, which is uſually ſtyled the determination of problems.

3. This, however, is not the only uſe to which imaginary expreſſions have been applied. When combined according to certain rules, they have been put to denote real quantities, and though they are in fact no more than marks of impoſſibility, they have been made the ſubjects of arithmetical operations; their ratios, their products, and their ſums, have been computed, and, what may ſeem ſtrange, juſt concluſions have in that way been deduced. Nevertheleſs, the name of reaſoning cannot be given to a proceſs into which no idea is introduced.

Accordingly geometry, which has its modes of reaſoning that correſpond to every other part of the algebraic calculus, has nothing ſimilar to the method we are now conſidering; for the arithmetic of mere characters can have no place in a ſcience which is immediately converſant with ideas.

But though geometry rejects this method of inveſtigation, it admits, on many occaſions, the concluſions derived from it, and has confirmed them by the moſt rigorous demonſtration. Here then is a paradox which remains to be explained. If the operations of this imaginary arithmetic are unintelligible, why are they not alſo uſeleſs? Is inveſtigation an art ſo mechanical, that it may be conducted by certain manual operations? or is truth ſo eaſily diſcovered, that intelligence is not neceſſary to give ſucceſs to our reſearches? Theſe are difficulties which it is of ſome importance to reſolve, and on which much attention has not hitherto been beſtowed. Two celebrated mathematicians, BERNOULLI and MACLAURIN, have indeed touched on this ſubject; but being more intent on applying their calculus, than on explaining the grounds of it, they have only ſuggeſted a ſolution of the difficulty, and one too by no means ſatisfactory. They alledge(a), that when imaginary expreſſions are put to denote real quantities, the imaginary characters involved in the different terms of ſuch expreſſions do then compenſate or deſtroy each other. But beſide that, the manner in which this compenſation is made, in expreſſions ever ſo little complicated, is extremely obſcure, if it be conſidered that an imaginary character is no more than a mark of impoſſibility, ſuch a compenſation becomes altogether unintelligible: for how can we conceive one impoſſibility removing or deſtroying another? Is not this to bring impoſſibility under the predicament of quantity, and to make it a ſubject of arithmetical computation? And are we not thus brought back to the very difficulty to be removed? Their explanation cannot of conſequence be admitted; but, on attempting another, it behoves us to obſerve, that a more extenſive application of this method, than had been made in their time, has now greatly facilitated the inquiry. We begin then with conſidering the manner in which the imaginary expreſſions, ſuppoſed to denote real quantities, are derived; and the caſes in which they prove uſeful for the purpoſes of inveſtigation.

(a) Op. J. BERN. tom. I. N° 70. MACLAUR. Flux. art. 699—763.

4. Let a be an arch of a circle of which the radius is unity, and let c be the number which has unity for its hyperbolic logarithm, then the ſine of the arch a, or

ſin. $a = \frac{c^{a\sqrt{-1}} - c^{-a\sqrt{-1}}}{2\sqrt{-1}}$; and coſ. $a = \frac{c^{a\sqrt{-1}} + c^{-a\sqrt{-1}}}{2}$. Theſe exponential and imaginary values of the ſine and coſine are already well known to geometers; and the inveſtigation of them, according to the received arithmetic of impoſſible quantities, may be as follows.

Let ſin. $a = z$, then $\dot{a} = \frac{\dot{z}}{\sqrt{1-z}}$. To bring this fluxion under ſuch a form that its fluent may be found by logarithms, both numerator and denominator are to be multiplied by $\sqrt{-1}$; then $\dot{a} = \sqrt{-1} \times \frac{\dot{z}}{\sqrt{z^2-1}}$, and (by form. 6. HARM. Men.) $a = \sqrt{-1} \times \log. \frac{z+\sqrt{z^2-1}}{\sqrt{-1}}$. Hence $\frac{a}{\sqrt{-1}}$, or $1 \times \frac{a}{\sqrt{-1}} = \log. \frac{z+\sqrt{z^2-1}}{\sqrt{-1}}$, and becauſe 1 is the log. of c, $c^{\frac{a}{\sqrt{-1}}} = \frac{z+\sqrt{z^2-1}}{\sqrt{-1}}$; wherefore, if both parts of the fractional index of c be multiplied by $\sqrt{-1}$, $c^{-a\sqrt{-1}} = \frac{z+\sqrt{z^2-1}}{\sqrt{-1}}$. Again, if the arch a be conſidered as negative, its ſine becomes alſo negative, and therefore $-a = \sqrt{-1} \times \log. \frac{-z+\sqrt{z^2-1}}{\sqrt{-1}}$, or, $-a\sqrt{-1} = -\log. \frac{-z+\sqrt{z^2-1}}{\sqrt{-1}}$, and $a\sqrt{-1} = \log. \frac{-z+\sqrt{z^2-1}}{\sqrt{-1}}$; whence alſo, $c^{a\sqrt{-1}} = \frac{-z+\sqrt{z^2-1}}{\sqrt{-1}}$. If from this equation the former be taken away, there remains $-\frac{2z}{\sqrt{-1}} = c^{a\sqrt{-1}} - c^{-a\sqrt{-1}}$, whence dividing by $2\sqrt{-1}$ we have $z = $ ſin. $a = \frac{c^{a\sqrt{-1}} - c^{-a\sqrt{-1}}}{2\sqrt{-1}}$. By adding together the equations a value of the coſine may be found in the ſame imaginary terms which were aſſigned above. Now by means of theſe expreſſions many theorems may be demonſtrated; it may, for example, be ſhewn, that if a and b are any two arches of a circle, of which the radius is unity, then $\text{ſin. } a \times \text{coſ. } b = \frac{\text{ſin. } \overline{a+b}}{2} + \frac{\text{ſin. } a-b}{2}$. For $\text{ſin. } a = \frac{c^{a\sqrt{-1}} - c^{-a\sqrt{-1}}}{2\sqrt{-1}}$, and $\text{coſ. } b = \frac{c^{b\sqrt{-1}} + c^{-b\sqrt{-1}}}{2}$, therefore, $\text{ſin. } a \times \text{coſ. } b = \frac{c^{\overline{a+b}\times\sqrt{-1}} - c^{\overline{-a-b}\times\sqrt{-1}} + c^{\overline{a-b}\times\sqrt{-1}} - c^{\overline{b-a}\times\sqrt{-1}}}{4\sqrt{-1}} = \frac{\text{ſin. } \overline{a+b}}{2} + \frac{\text{ſin. } \overline{a-b}}{2}$.

5. Now it may be obſerved, that the imaginary value which has been found for ſin. a was obtained by bringing a fluxion, properly belonging to the circle, under the form of one belonging to the hyperbola. It may, therefore, be worth while to inquire, whether a ſimilar expreſſion may not be derived from the hyperbola itſelf.

Let BAD be a rectangular hyperbola (fig. 2.) of which the center is C, and the ſemi-tranverſe axis AC $= 1$; let B be any point in the hyperbola, join BC, and let BE be an ordinate to the tranſverſe axis. Then, if the ſector ACB $= \frac{1}{2}a$, and BE $= z$, it is known that $a = \frac{\dot{z}}{\sqrt{1+z^2}}$; whence $a = \log. z + \sqrt{1+z^2}$, and $c^a = z + \sqrt{1+z^2}$. But if the ſector be taken on the other ſide of the tranſverſe axis, a and z

become negative, and $c^{-a} = -z + \sqrt{1+z^2}$. Hence $z = \frac{c^a - c^{-a}}{2}$; in like manner the abſciſs belonging to ACB, that is CE $= \frac{c^a + c^{-a}}{2}$. Theſe values of the ordinates and abſciſſæ differ in nothing from thoſe of the ſines and coſines already found, except in being free from impoſſible quantities; for it is evident, that the quantity a is related in the ſame manner both to the circular and hyperbolic ſectors. If now ord. a and abſ. b denote the ordinate and abſciſs belonging to the ſectors $\frac{1}{2}a$, $\frac{1}{2}b$ reſpectively, ord. a × abſ. $b = \frac{c^a - c^{-a}}{2} \times \frac{c^b + c^{-b}}{2} = \frac{c^{a+b} - c^{-a-b} + c^{a-b} - c^{b-a}}{4} = \frac{\text{ord. } \overline{a+b}}{2} + \frac{\text{ord. } \overline{a-b}}{2}$.

6. The concluſions in both the foregoing caſes are perfectly coincident, and the methods by which they have been obtained are ſimilar; though with this difference between them, that in the firſt all the ſteps are unintelligible, but in the laſt ſignificant. If then, notwithſtanding a difference which might be expected ſo materially to affect their concluſions, they have been equally ſucceſsful in the diſcovery of truth, it can be aſcribed only to the analogy which takes place between the ſubjects of inveſtigation; an analogy ſo cloſe, that every property belonging to the one may, with certain reſtrictions, be transferred to the other. Accordingly,

every imaginary expreſſion, which has been found to belong to the circle in the preceding calculation, is by the ſubſtitution of real for impoſſible quantities, or of $\sqrt{1}$ for $\sqrt{-1}$, converted into a propoſition which holds of the hyperbola. The operations, therefore, performed with the imaginary characters, though deſtitute of meaning themſelves, are yet notes of reference to others which are ſignificant. They point out indirectly a method of demonſtrating a certain property of the hyperbola, and then leave us to conclude from analogy that the ſame property belongs alſo to the circle. All that we are aſſured of by the imaginary inveſtigation is, that its concluſion may, with all the ſtrictneſs of mathematical reaſoning, be proved of the hyperbola; but if from thence we would transfer that concluſion to the circle, it muſt be in conſequence of the principle which has been juſt now mentioned. The inveſtigation, therefore, reſolves itſelf ultimately into an argument from analogy; and, after the ſtricteſt examination, will be found without any other claim to the evidence of demonſtration. Had the foregoing propoſition been proved of the hyperbola only, and afterwards concluded to hold of the circle, merely from the affinity of the curves, its certainty would have been preciſely the ſame as when a proof is made out by the intervention of imaginary ſymbols.

8. Though it might readily be concluded, that the ſame principle on which the foregoing inveſtigation has been found to proceed, extends itſelf to all thoſe in which imaginary expreſſions are put to denote real quantities, it may yet be proper to make trial of its application in ſome other inſtances.

Let AB, AC, AD, AE (fig. 3.) be any arches of a circle in arithmetical progreſſion, and let m be their number; it is required to find the ſum of the ſines BC, CH, &c. of thoſe arches. Let the radius AF$=1$, AB$=a$, and the common difference of the arches, or BC$=x$: the ſum of the ſeries ſin. $a+$ſin. $\overline{a+x}$ + ſin. $\overline{a+2x}$ + ... (m) is to be found. Now, becauſe ſin. $a=\frac{c^{a\sqrt{-1}}-c^{-a\sqrt{-1}}}{2\sqrt{-1}}$, and ſin. $a+x=\frac{c^{\overline{a+x}\times\sqrt{-1}}-c^{\overline{-a-x}\times\sqrt{-1}}}{2\sqrt{-1}}$, &c.; the ſeries ſin. a + ſin. $a+x$ + ſin. $a+2x$ . . . . . (m) $=\frac{c^{a\sqrt{-1}}}{2\sqrt{-1}}\times$ $\overline{1+c^{x\sqrt{-1}}+c^{2x\sqrt{-1}}}$ . . . . . $(m)-\frac{c^{a\sqrt{-1}}}{2\sqrt{-1}}\times$ $\overline{1+c^{-x\sqrt{-1}}+c^{-2x\sqrt{-1}}}$ . . . . . (m). But theſe ſeries are both geometrical progreſſions, and the ſum of the firſt is $\frac{c^{a\sqrt{-1}}}{2\sqrt{-1}}\times\frac{1-c^{mx\sqrt{-1}}}{1-c^{x\sqrt{-1}}}$; and of the ſecond, $\frac{c^{-a\sqrt{-1}}}{2\sqrt{-1}}\times\frac{1-c^{-mx\sqrt{-1}}}{1-c^{-x\sqrt{-1}}}$. The ſum of the propoſed ſeries therefore

$$=\frac{c^{a\sqrt{-1}}}{2\sqrt{-1}}\times\frac{1-c^{mx\sqrt{-1}}}{1-c^{x\sqrt{-1}}}-\frac{c^{-a\sqrt{-1}}}{2\sqrt{-1}}\times\frac{1-c^{-mx\sqrt{-1}}}{1-c^{-x\sqrt{-1}}}=\frac{1}{2\sqrt{-1}}\times$$

$$\frac{c^{a\sqrt{-1}}-c^{\overline{a+mx}\times\sqrt{-1}}-c^{\overline{a-x}\times\sqrt{-1}}+c^{\overline{a+mx-x}\times\sqrt{-1}}}{1-c^{x\sqrt{-1}}-c^{-x\sqrt{-1}}+1}+\frac{1}{2\sqrt{-1}}\times$$

$$-c^{-a-\sqrt{-1}}$$

$$\frac{-c^{-a\sqrt{-1}}+c^{-\overline{a-mx}\times\sqrt{-1}}+c^{-\overline{a+x}\times\sqrt{-1}}-c^{-\overline{a-mx+x}\times\sqrt{-1}}}{1-c^{x\sqrt{-1}}-c^{-x\sqrt{-1}}+1};$$

in which expreſſion, if the ſines be ſubſtituted for their imaginary values, we have

$$\frac{\text{ſin. }a-\text{ſin. }\overline{a+mx}-\text{ſin. }\overline{a-x}+\text{ſin. }\overline{a+mx-x}}{2\times\overline{1-\text{coſ. }x}}=$$

$\text{ſin. }a+\text{ſin. }\overline{a+x}+\text{ſin. }\overline{a+2x}\ \ldots\ldots(m)$. Q. E. I.

When AB=BC, or $a=x$, the propoſed ſeries becomes $\text{ſin. }x+\text{ſin. }2x+\text{ſin. }3x\ \ldots\ldots(m)$, and its value $=\dfrac{\text{ſin }x-\text{ſin. }\overline{m+1}\times x+\text{ſin. }mx}{2\times\overline{1-\text{coſ. }x}}$.

In like manner it will be found, that the ſum of the coſines of the ſame arches, or $\text{coſ. }a+\text{coſ. }\overline{a+x}+\text{coſ. }\overline{a+2x}+\ldots\ldots(m)=\dfrac{\text{coſ. }a-\text{coſ. }\overline{a+mx}-\text{coſ. }\overline{a-x}+\text{coſ. }\overline{a+mx-x}}{2\times\overline{1-\text{coſ. }x}}$: and when $a=x$, $\text{coſ. }x+\text{coſ. }2x+\text{coſ. }3x\ \ldots\ldots(m)=\dfrac{\text{coſ. }mx-\text{coſ. }\overline{m+1}\times x}{2\times\overline{1-\text{coſ. }x}}-\dfrac{1}{2}$.

9. To ſolve the ſame problem, in the caſe of the hyperbola, we muſt follow the ſteps which have been traced out by theſe imaginary operations. Let ABE be an equilateral hyperbola (fig. 4.) of which the center is F, and the tranſverſe axis AF=1; let ABF, ACF, ADF, &c. be any ſectors in arithmetical progreſſion, and let m be their number; it is required to find the ſum of all the ordinates BG, CH, DK, &c. belonging thoſe ſectors. Let the ſector AFB $=\frac{1}{2}a$, and the ſector BFC, which is the

common

common difference of the ſectors, $=\frac{1}{2}x$: then BG, or ord. $a = \frac{c^{a} - c^{-a}}{2}$, and CH, or ord. $\overline{a+x} = \frac{c^{a+x} - c^{-a-x}}{2}$, by art. 5. Therefore the ſeries of ordinates, that is, $\text{BG}+\text{CH}+\text{DK}+ \ldots\ldots (m) = \frac{c^{a}}{2} \times \overline{1 + c^{x} + c^{2x} + \ldots\ldots (m)} - \frac{e^{-a}}{2} \times \overline{1 + c^{-x} + c^{-2x} + \ldots\ldots} (m) = \frac{c^{a}}{2} \times \frac{1 - c^{mx}}{1 - c^{x}} - \frac{c^{-a}}{2} \times \frac{1 - c^{-mx}}{1 - c^{-x}}$ $= \frac{1}{2} \times \frac{c^{a} - c^{a+mx} - c^{a-x} + c^{a+mx-x} - c^{-a} + c^{-a-mx} + c^{-a-x} - c^{-a-mx+x}}{1 - c^{x} - c^{-x} + 1} =$ $\frac{\text{ord. } a - \text{ord. } \overline{a+mx} - \text{ord. } \overline{a-x} + \text{ord. } \overline{a+mx-x}}{2 \times \overline{1 - \text{abſ. } x}}$. When $a = x$, ord. x + ord. $2x$ + ord. $3x + \ldots\ldots (m) =$ $\frac{\text{ord. } x - \text{ord. } \overline{m+1} \times x + \text{ord. } mx}{2 \times \overline{1 - \text{abſ. } x}}$.

In like manner it is proved, that the ſum of the abſciſſæ, that is, $\text{FG}+\text{FH}+\text{FK}+ \ldots\ldots (m) =$ $\frac{\text{abſ } a - \text{abſ } \overline{a+mx} - \text{abſ } \overline{a-x} + \text{abſ. } \overline{a+mx-x}}{2 \times \overline{1 - \text{abſ. } x}}$; and when $a = x$, this expreſſion becomes $\frac{\text{abſ. } mx - \text{abſ. } \overline{m+1} \times x}{2 \times \overline{1 - \text{abſ. } x}} - \frac{1}{2}$.

10. The coincidence of the theorems deduced in the two laſt articles is obvious at firſt ſight, and if the methods by which they have been obtained be compared, it will appear, that the imaginary operations in the one caſe were of no uſe but as they adumbrated the real demonſtration, which took place in the other. This will be rendered more evident by conſidering that the reſolution of the ſeries of hyperbolic ordinates, into two others

of continual proportionals, can be exhibited geometrically. For, from the points A, B, C, and D, let AM, BN, CO, DP, be drawn at right angles to the aſſymptote FP; let GB produced meet FP in Q, and let BR be perpendicular to the conjugate axis FR. Then, becauſe the triangles FRS, FMA, are equiangular, AF : FM :: FS : FR; hence $\mathrm{FR} = \frac{\mathrm{FM}}{\mathrm{FA}} \times \mathrm{FS} = \frac{\mathrm{FM}}{\mathrm{FA}} \times \overline{\mathrm{FN} - \mathrm{NB}}$. For the ſame reaſon $\mathrm{CH} = \frac{\mathrm{FM}}{\mathrm{FA}} \times \overline{\mathrm{FO} - \mathrm{OC}}$, and $\mathrm{DK} = \frac{\mathrm{FM}}{\mathrm{FA}} \times \overline{\mathrm{FP} - \mathrm{PD}}$. Therefore, $\mathrm{BG} + \mathrm{CH} + \mathrm{DK} = \frac{\mathrm{FM}}{\mathrm{FA}} \times \overline{\mathrm{FN} + \mathrm{FO} + \mathrm{FP}} - \frac{\mathrm{FM}}{\mathrm{FA}} \times \overline{\mathrm{BN} + \mathrm{CO} + \mathrm{DP}}$; now, FN, FO, FP, are continual proportionals, and ſo alſo are BN, FO, FP, becauſe the ſectors FBC, FCD, are equal. But in the circle no ſuch reſolution of the propoſed ſeries of ſines can take place, that ſeries being ſubject to alternate increaſe and diminution; on which account it is, that imaginary characters enter into the exponential value of the ſine. Thoſe characters are therefore ſo far from compenſating each other in the preſent caſe, as they ought to do, on the ſuppoſition of BERNOULLI and MACLAURIN, that they manifeſtly ſerve as marks of impoſſibility. There remains, of conſequence, the affinity between circular arches and hyperbolic areas, or between the meaſures of angles and of ratios, as the only principle on which the imaginary inveſtigation can proceed. It need ſcarcely be obſerved, that the exponential value of the hyperbolic ordinate may be deduced from what has been proved in this article.

11. But as the arithmetic of impoſſible quantities is no where of greater uſe than in the inveſtigation of fluents, it is of conſequence to inquire, whether the preceding theory extends alſo to that application of it.

Let it then be required to find the fluent of the equation $\frac{\ddot{y}}{\dot{z}^2} \mp a^2 y = Q$, where Q denotes any function whatever of x. For this purpoſe, the following lemma is premiſed: let x be any arch, and p any flowing quantity; then, if the ſign $\int$, be taken to denote the fluent of the quantity to which it is prefixed, ſin. $x\int\dot{p}$ coſ. x − coſ. $x\int\dot{p}$ ſin. $x =$ $\frac{c^{x\sqrt{-1}}}{2\sqrt{-1}}\int\dot{p}c^{-x\sqrt{-1}} - \frac{c^{-x\sqrt{-1}}}{2\sqrt{-1}}\int\dot{p}c^{x\sqrt{-1}}$; or if $\frac{1}{2}x$ be a hyperbolic ſector, ord. $x\int\dot{p}$ abſ. x − abſ. $x\int\dot{p}$ ord. $x =$ $\frac{c^x}{2}\int\dot{p}c^{-x} - \frac{c^{-x}}{2}\int\dot{p}c^x$.

Becauſe ſin. $x\int\dot{p}$ coſ. $x = \frac{c^{x\sqrt{-1}} - c^{-x\sqrt{-1}}}{2\sqrt{-1}}\int\dot{p} \times \frac{c^{x\sqrt{-1}} + c^{-x\sqrt{-1}}}{2}$, by ſeparating the terms we have ſin. $x\int\dot{p}$ coſ. $x =$ $\frac{c^{x\sqrt{-1}}}{4\sqrt{-1}}\int\dot{p}c^{x\sqrt{-1}} + \frac{c^{x\sqrt{-1}}}{4\sqrt{-1}}\int\dot{p}c^{-x\sqrt{-1}} - \frac{c^{-x\sqrt{-1}}}{4\sqrt{-1}}\int\dot{p}c^{x\sqrt{-1}} - \frac{c^{-x\sqrt{-1}}}{4\sqrt{-1}}\int\dot{p}c^{-x\sqrt{-1}}$, for the ſame reaſon − coſ. $x\int\dot{p}$ ſin. $x =$ $-\frac{c^{x\sqrt{-1}}}{4\sqrt{-1}}\int\dot{p}c^{x\sqrt{-1}} + \frac{c^{x\sqrt{-1}}}{4\sqrt{-1}}\int\dot{p}c^{-x\sqrt{-1}} - \frac{c^{-x\sqrt{-1}}}{4\sqrt{-1}}\int\dot{p}c^{x\sqrt{-1}} +$

$\frac{c^{-x\sqrt{-1}}}{4\sqrt{-1}}\int \dot{p} c^{-x\sqrt{-1}}$. Wherefore, by collecting the ſum of all the terms, we have ſin. $x\int\dot{p}$ coſ. x − coſ. $x\int\dot{p}$ ſin. x = $\frac{c^{x\sqrt{-1}}}{2\sqrt{-1}}\int\dot{p}c^{-x\sqrt{-1}} - \frac{c^{-x\sqrt{-1}}}{2\sqrt{-1}}\int\dot{p}c^{x\sqrt{-1}}$.

The demonſtration in the caſe of the hyperbola is free from imaginary expreſſions; but, in other reſpects, is exactly ſimilar to that which has now been given in the caſe of the circle.

12. Let the co-efficient of y in the propoſed equation be firſt ſuppoſed negative, that is, let $\frac{\ddot{y}}{\dot{x}^2} - a^2y = Q$, and if we multiply by $c^{nx}\dot{x}$, n being a conſtant but indeterminate quantity, it becomes $\frac{c^{nx}\ddot{y}}{\dot{x}} - a^2c^{nx}y\dot{x} = c^{nx}Q\dot{x}$. Let $c^{nx} \times \overline{\frac{A\dot{y}}{\dot{x}} - By}$ be aſſumed for the fluent, A and B being indeterminate, and let its fluxion be taken, then,

$$\frac{Ac^{nx}\ddot{y}}{\dot{x}} + nAc^{nx}\dot{y} - nBc^{nx}y\dot{x} = c^{nx}Q\dot{x}.$$
$$- Bc^{nx}\dot{y}.$$

Hence, by comparing the terms, we get $A = 1$, $nA - B = 0$, $nB = a^2$; therefore, $n = \pm a$, and $B = \pm a$: for n and B let the value $+a$ be ſubſtituted, and for A, its value, unity; and the aſſumed equation becomes $\overline{\frac{\dot{y}}{\dot{x}} - ay} \times c^{ax} = \int c^{ax}Q\dot{x}$, or $\frac{\dot{y}}{\dot{x}} - ay = c^{-ax}\int c^{ax}Q\dot{x}$. Let this equation be multiplied by $c^{mx}\dot{x}$, m being indeterminate as before, and

$c^{mx}\dot{y}-ac^{mx}y\dot{x}=c^{\overline{m-a}\times x}\dot{x}\int c^{ax}Q\dot{x}$. The fluent of the firſt member of this equation is evidently of the form $Dc^{mx}y$, the fluxion of which, *viz.* $Dc^{mx}\dot{y}+Dmc^{mx}y\dot{x}$ being compared with the former gives $D=1$, and $m=-a$; wherefore, $c^{-ax}y=\int c^{-2ax}\dot{x}\int c^{ax}Q\dot{x}$, or $y=c^{ax}\times\int c^{-2ax}\dot{x}\int c^{ax}Q\dot{x}$. Let $c^{ax}Q\dot{x}=\dot{z}$, and $c^{-2ax}\dot{x}=\dot{v}$; then $\int c^{-2ax}\dot{x}\int c^{ax}Q\dot{x}=\int z\dot{v}=zv-\int v\dot{z}$; but $v=\frac{1}{2a}-\frac{c^{-2ax}}{2a}$, ſuppoſing that v and x vaniſh at the ſame time; therefore $vz-\int v\dot{z}=\frac{1}{2a}\int c^{ax}Q\dot{x}-\frac{c^{-2ax}}{2a}\int c^{ax}Q\dot{x}-\frac{1}{2a}\int c^{ax}Q\dot{x}+\frac{1}{2a}\int c^{-ax}Q\dot{x}=\frac{1}{2a}\int c^{-ax}Q\dot{x}-\frac{c^{-2ax}}{2a}\int c^{ax}Q\dot{x}$. Hence $y=\frac{c^{ax}}{2a}\int c^{-ax}Q\dot{x}-\frac{c^{-ax}}{2a}\int c^{ax}Q\dot{x}$. This value of y is ſufficient for the conſtruction of the fluent, becauſe the quantities $\int c^{-ax}Q\dot{x}$, and $\int c^{ax}Q\dot{x}$ depend on the quadrature of the hyperbola; but if we would introduce into it the ordinates and abſciſſes of that curve, we need only have recourſe to the foregoing lemma, from which it appears, that $y=\frac{1}{a}$ ord. $ax\int Q\dot{x}$ abſ. $ax-\frac{1}{a}$ abſ. $ax\int Q\dot{x}$ ord. ax.

13. Let the co-efficient of y be now ſuppoſed affirmative, or let $\frac{\ddot{y}}{\dot{x}^2}+a^2y=Q$. In this caſe imaginary expreſſions are introduced into the fluent, and the conſtruction by

the hyperbola becomes impoſſible. For we have then, $n=\pm a\sqrt{-1}$, from which, by proceeding as above, we get $y=\frac{c^{ax-\sqrt{-1}}}{2a\sqrt{-1}}\int c^{-ax\sqrt{-1}}Q\dot{x}-\frac{c^{-ax}}{2a\sqrt{-1}}\int c^{ax\sqrt{-1}}Q\dot{x}$; hence alſo, by the lemma, $y=$ ſin. $ax\int Q\dot{x}$ coſ. $ax-$ coſ. $ax\int Q\dot{x}$ ſin. ax. Here the quantities, $\int Q\dot{x}$ coſ. ax, and $\int Q\dot{x}$ ſin. ax, are aſſignable by the quadrature of the circle, in the ſame manner as $\int Q\dot{x}$ abſ. ax, and $\int Q\dot{x}$ ord. ax, by the quadrature of the hyperbola; but the method of inveſtigating them, though an illuſtration of the principles which we have laid down, is too well known to need to be inſerted here. In like manner might the fluents of innumerable fluxionary equations, comprehended under the general form $Q=y+\frac{a\dot{y}}{\dot{x}}+\frac{b\ddot{y}}{\dot{x}^2}+\frac{d\dddot{y}}{\dot{x}^3}+$ &c. be deduced, and all of them would tend to prove that the arithmetic of impoſſible quantities is no more than a method of tracing the analogy between the meaſures of ratios and of angles. M. M. EULER(b) and D'ALEMBERT(c) were the firſt to integrate ſuch equations as the preceding, and the method employed here differs from theirs only by being better adapted to illuſtrate the principle which is common to them all.

(b) Nov. Com. Petrop. tom. III.

(d) Theorie de la Lune.

14. The forms in the *Harmonia Menſurarum* might alſo be brought to confirm this theory; but, without accumulating inſtances any farther, it may be ſufficient to remark two conſequences that follow from it: 1. That the only caſes in which imaginary expreſſions may be put to denote real quantities, are thoſe in which the meaſures of ratios or of angles are concerned. 2. That the property of either of thoſe meaſures, ſo inveſtigated, might have been inferred from analogy alone. Now both theſe concluſions are agreeable to experience. It does not appear, that any inſtance has yet occurred where imaginary characters ſerve to expreſs real quantities, if circular arches or hyperbolic areas are not the ſubjects of inveſtigation; and if the concluſion obtained may not be transferred from the one to the other, by a mere ſubſtitution of correſponding magnitudes; that is, of ſines for ordinates, coſines for abſciſſes, and circular arches for the doubles of hyperbolic ſectors. The affinity between the circle and hyperbola is not however ſo cloſe, but that it is ſubject to certain limitations, from conſidering which, the truth of what is here aſſerted will be rendered more evident.

1. Any propoſition demonſtrated of hyperbolic ſectors may be transferred to circular arches by ſubſtitution alone, without any change in the ſigns, when only

abſciſſæ and their products enter into the enunciation, and converſely. Thus abſ. $a \times$ abſ. $b = \frac{\text{abſ.}\ \overline{a+b}}{2} + \frac{\text{abſ.}\ \overline{a-b}}{2}$; and coſ. $a \times$ coſ. $b = \frac{\text{coſ.}\ \overline{a+b}}{2} + \frac{\text{coſ.}\ \overline{a-b}}{2}$. The ſame holds when the ſimple power of the ordinate is combined with any power whatever of the abſciſs: ſo in the theorems of art. 5. and 4. ord. $a \times$ abſ. $b = \frac{\text{ord.}\ \overline{a+b}}{2} + \frac{\text{ord.}\ \overline{a-b}}{2}$; and ſin. $a \times$ coſ. $b = \frac{\text{ſin.}\ \overline{a+b}}{2} + \frac{\text{ſin.}\ \overline{a-b}}{2}$.

2. When an expreſſion containing any property of hyperbolic ſectors, involves in it the rectangle of two ordinates, the value of that rectangle muſt have a contrary ſign, when a tranſition is made to the circle. Thus ord. $a \times$ ord. $b = \frac{\text{abſ.}\ \overline{a+b}}{2} - \frac{\text{abſ.}\ \overline{a-b}}{2}$; but ſin. $a \times$ ſin. $b = -\frac{\text{coſ.}\ \overline{a+b}}{2} + \frac{\text{coſ.}\ \overline{a-b}}{2}$. The difference which according to this rule is found between the powers of ordinates and of ſines may be ſeen in the following examples. If $\frac{1}{2}x$ denote any hyperbolic ſector, then, by involving $\frac{c^x - c^{-x}}{2}$, and again ſubſtituting for the exponential quantities as in art. 5. we have,

$$\overline{\text{ord.}\ x}^2 = \frac{\text{abſ.}\ 2x - 1}{2};$$

$$\overline{\text{ord.}\ x}^3 = \frac{\text{ord.}\ 3x - 3\ \text{ord.}\ x}{4};$$

$$\overline{\text{ord.}\ z}^4 = \frac{\text{abſ.}\ 4x - 4\ \text{abſ.}\ 2x + 3}{8};$$

$\overline{\text{ord.}\,x}^{5} = \frac{\text{ord.}\,5x - 5\,\text{ord.}\,3x + 10\,\text{ord.}\,x}{16}$; and univerſally, if n be any number; a the co-efficient of the ſecond term of a binomial raiſed to the power n, b the co-efficient of the third, &c. and p the greateſt co-efficient: when n is an even number,

$$\overline{\text{ord.}\,x}^{n} = \frac{\text{abſ.}\,nx - a\,\text{abſ.}\,\overline{n-2}\times x + b\,\text{abſ.}\,\overline{n-4}\times x \ldots\ldots \mp p}{2^{n-1}} \pm \frac{p}{2^{n}};$$

but when n is an odd number,

$$\overline{\text{ord.}\,x}^{n} = \frac{\text{ord.}\,nx - a\,\text{ord.}\,\overline{n-2}\times x + b\,\text{ord.}\,\overline{n-4}\times x \ldots\ldots \mp p\,\text{ord.}\,x}{2^{n-1}}.$$

If now x denote an arch of a circle, by ſubſtituting and changing the ſigns as oft as $\overline{\text{ord.}\,x}^{2}$ occurs in any of the preceding expreſſions, we get

$$\overline{\text{ſin.}\,x}^{2} = \frac{1 - \text{coſ.}\,2x}{2};$$

$$\overline{\text{ſin.}\,x}^{3} = \frac{3\,\text{ſin.}\,x - \text{ſin.}\,3x}{4}:$$

$$\overline{\text{ſin.}\,x}^{4} = \frac{3 - 4\,\text{coſ.}\,2x + \text{coſ.}\,4x}{8};$$

$\overline{\text{ſin.}\,x}^{5} = \frac{10\,\text{ſin.}\,x - 5\,\text{ſin.}\,3x + \text{ſin.}\,5x}{16}$: and univerſally, if n be any number, p the greateſt co-efficient of a binomial raiſed to the power n, A the co-efficient next leſs than p, B the co-efficient next leſs than A, and ſo on: when n is an even number,

$$\overline{\text{ſin.}\,x}^{n} = \frac{\frac{1}{2}p - \text{A coſ.}\,2x + \text{B coſ.}\,4x - \text{\&c.}}{2^{n-1}};$$

but when n is an odd number,

$$\overline{\text{ſin.}\,x}^{n} = \frac{p\,\text{ſin.}\,x - \text{A coſ.}\,3x + \text{B coſ.}\,5x - \text{\&c.}}{2^{n-1}}.$$

Theſe ſeries differ from the former only in the ſigns, and the arrangement of the terms; and when either n, or $n-1$, is diviſible by 4, the ſigns remain the ſame in both.

16. The reaſon of the foregoing rule for changing the ſigns is, that the rectangle under two ordinates to the hyperbola is always expreſſed by the difference of two abſciſſæ: and that if from the abſciſs belonging to a greater ſector, be ſubtracted the abſciſs belonging to a leſs, the remainder will be affirmative; whereas, if from the coſine of a greater arch be ſubtracted the coſine of a leſs, the remainder will be negative. Therefore, that the rectangles, expreſſed by theſe remainders, may have the ſame ſign, in both caſes, the ſigns of the remainders muſt be different.

It appears then, that the ſecond rule, as well as the firſt, is founded on the principle of analogy when taken with the neceſſary limitations, and it is likewiſe evident from the inſtances which have been produced, that thoſe rules lead to the very ſame concluſions which are obtained from the imaginary values of the ſine and coſine.

There are, however, inſtances in which the analogy between the circular and hyperbolic areas being wholly interrupted, neither the foregoing rules, nor any of the ſame kind, can be applied; but this occaſions no ambiguity,

guity for the conſtruction required in ſuch caſes is by its nature reſtricted to one of the curves only. Of this kind is the Coteſian theorem, which requires the whole circle to be divided into a given number of equal parts, and therefore cannot be extended to the hyperbola where a ſimilar diviſion is impoſſible. Others of a like nature may be derived from the general theorems already inveſtigated; for the circle, by returning into itſelf, often reduces them to a ſimplicity to which there is nothing analogous in the hyperbola. Many examples of this might be adduced, but the two following may ſuffice.

1. Let ABCDE (fig. 5.) be a regular polygon inſcribed in a circle, and let m be the number of its ſides; it is required to find the ſum of the lines FA, FB, FC, &c. drawn from any point F in the circumference, to all the angles of the polygon. By the method which in art. 8. was employed to obtain the ſum of the ſines of a ſeries of arches in arithmetical progreſſion, it will be found, that the ſum of the chords of the arches a, $\overline{a+x}$, $\overline{a+2x}$, ... (m), that is, (making FA$=a$, and AB$=x$) the ſum of the chords of the arches FA, FB, FC, &c. $= \frac{\text{cho.}\, a - \text{cho.}\, \overline{a+mx} - \text{cho.}\, \overline{a-x} + \text{cho.}\, \overline{a+mx-x}}{2 \times \overline{1 - \text{coſ.}\, \frac{1}{2}x}}$; but, in the preſent caſe, mx is equal to the circumference, and therefore $-\text{cho.}\, \overline{a+mx} = +\text{cho.}\, a$ (the chord of an arch greater than

the circumference being negative); and, for the ſame reaſon, $\text{cho.}\ \overline{a+mx-x}=-\text{cho.}\ \overline{a-x}=+\text{cho.}\ \overline{x-a}$. Hence the general expreſſion becomes $\frac{\text{cho.}\ a+\text{cho.}\ \overline{x-a}}{1-\text{coſ.}\ \frac{1}{2}x}=\mathrm{FA}+\mathrm{FB}+\mathrm{FC}+\ldots\ldots(m)$. If therefore GK be drawn from the center, biſecting the chord AB in H, and meeting the circumference in K, the ſum of the chords, that is, $\mathrm{FA}+\mathrm{FB}+\mathrm{FC}+\mathrm{FD}+\mathrm{FE}=\frac{\mathrm{AF}+\mathrm{FB}}{\mathrm{FK}}\times\mathrm{GK}$.

2. Let n be an even number, the reſt remaining as above, and let it be required to find the ſum of the n powers of the chords, that is, the ſum of $\overline{\mathrm{FA}}^{n}+\overline{\mathrm{FB}}^{n}+\overline{\mathrm{FC}}^{n}\ldots\ldots(m)$. By reaſoning, as in the caſe of the ſines, it will appear that, if p be the greateſt co-efficient of a binomial raiſed to the power n; A the co-efficient next leſs than p; B the co-efficient next leſs a. and ſo on, then,

$$\overline{\text{cho.}\ a}^{\,n}=p-2\mathrm{A}\ \text{coſ.}\ a+2\mathrm{B}\ \text{coſ.}\ 2a+2\mathrm{D}\ \text{coſ.}\ 3a+\ \&\text{c.}$$

$$\overline{\text{cho.}\ \overline{a+x}}^{\,n}=p-2\mathrm{A}\ \text{coſ.}\ \overline{a+x}+2\mathrm{B}\ \text{coſ.}\ 2\times\overline{a+x}+2\mathrm{D}\ \text{coſ.}\ 3\times\overline{a+x},\ \&\text{c.}$$

$$\overline{\text{cho.}\ \overline{a+2x}}^{\,n}=p-2\mathrm{A}\ \text{coſ.}\ \overline{a+x}+2\mathrm{B}\ \text{coſ.}\ 2\times\overline{a+2x}+2\mathrm{D}\ \text{coſ.}\ 3\times\overline{a+2x},\ \&\text{c.}$$

&c.

Each of theſe vertical columns is to be continued downward, till the number of terms be equal to m, and therefore the ſum of the ſecond is mp. The ſum of the third, or of $-2\mathrm{A}\times\overline{\text{coſ.}\ a+\text{coſ.}\ \overline{a+x}+\text{coſ.}\ \overline{a+2x}\ \ldots\ldots}\ (m)$, by art. 8. is $-2\mathrm{A}\times\frac{\text{coſ.}\ a-\text{coſ.}\ \overline{a+mx}-\text{coſ.}\ \overline{a-x}+\text{coſ.}\ \overline{a+mx-x}}{2\times\overline{1-\text{coſ.}\ x}}=$ (becauſe $mx=$ the circumference)

—A×

$-\mathrm{A}\times\frac{\text{coſ.}\,a-\text{coſ.}\,a-\text{coſ.}\,\overline{a-x}+\text{coſ.}\,\overline{a-x}}{1-\text{coſ.}\,x}=0$. In like manner do the ſums of all the ſubſequent columns vaniſh; and therefore, $\text{cho.}\,a+\text{cho.}\,\overline{a+x}+\text{cho.}\,\overline{a+2x}\ \ldots\ldots\ (m)=mp$. But when n is an even number, $p=\frac{\frac{1}{2}n+1}{\frac{1}{2}n-1}\times\frac{\frac{1}{2}n+2}{\frac{1}{2}n-2}\ \ldots\ldots\ \times\frac{n}{\frac{1}{2}n}$ $=\frac{1.3.5.7\ \ldots\ldots\ \overline{n-1}}{1.2.3.4\ \ldots\ldots\ \frac{1}{2}n}\times 2^{\frac{1}{2}n}$. If therefore the radius be put $=r$, and the expreſſion made homogeneous, we have $\overline{\mathrm{FA}}^{\,n}+\overline{\mathrm{FB}}^{\,n}+\overline{\mathrm{FC}}^{\,n}\ \ldots\ldots\ (m)=m\times\frac{1.3.5\,7\ \ldots\ \overline{n-1}}{1.2.3.4.\ \ldots\ldots\ \frac{1}{2}n}\times 2^{\frac{1}{2}n}r^{n}$.

Q. E. I.

This laſt coincides with the forty-firſt of the curious and difficult propoſitions publiſhed by Dr. STEWART, under the title of general theorems: it is given there without a demonſtration, but appears plainly to have been inveſtigated, in a manner altogether rigorous, by that profound geometer. It may therefore be regarded as one of the inſtances, in which the concluſions of this imaginary arithmetic are verified by the geometrical analyſis.

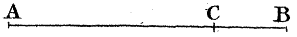
17. The two foregoing propoſitions being confined to the circle, and yet having been inveſtigated by the help of imaginary expreſſions, may, at firſt ſight, ſeem exceptions to the rule, which we have been endeavouring to eſtabliſh. But it needs only to be remarked, that they are particular caſes of certain theorems belonging both

to

to the circle and hyperbola, and that it was into the inveſtigation of thoſe theorems, that the imaginary expreſſions were introduced.

The concluſions therefore from the whole are theſe: that imaginary expreſſions are never of uſe in inveſtigation but when the ſubject is a property common to the meaſures both of ratios and of angles; that they never lead to any conſequence which might not be drawn from the affinity between thoſe meaſures; and that they are indeed no more than a particular method of tracing that affinity. The deductions into which they enter are thus reduced to an argument from analogy, but the force of them is not diminiſhed on that account. The laws to which this analogy is ſubject; the caſes in which it is perfect, in which it ſuffers certain alterations, and in which it is wholly interrupted, are capable, as may be concluded from the ſpecimens above, of being preciſely aſcertained. Supported on ſo ſure a foundation, the arithmetic of impoſſible quantities will always remain an uſeful inſtrument in the diſcovery of truth, and may be of ſervice when a more rigid analyſis can hardly be applied. For this reaſon, many reſearches concerning it, which in themſelves might be deemed abſurd, are nevertheleſs not deſtitute of utility. M. BERNOULLI has found, for example, that if r be the radius of a circle, the circumference

Fig: 1.



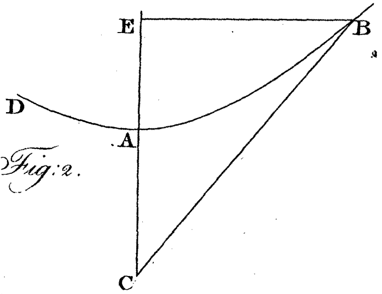


Fig: 2.

Fig: 3.

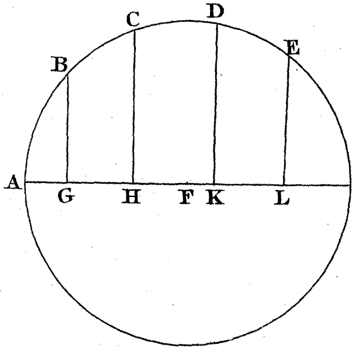


Fig: 4.

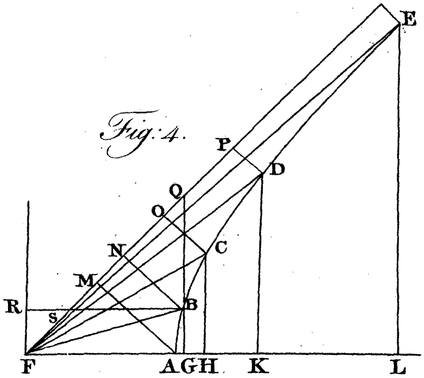
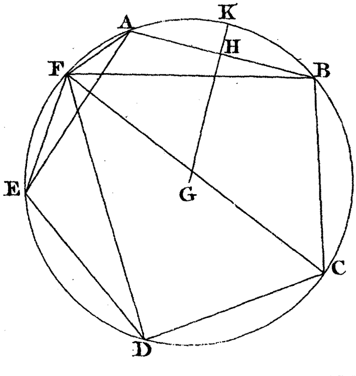


Fig: 5.



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$= \frac{4 \log. \sqrt{-1}}{\sqrt{-1}} r$; and the ſame may be deduced from art. 4. Conſidered as a quadrature of the circle, this imaginary theorem is wholly inſignificant, and would deſervedly paſs for an abuſe of calculation; at the ſame time we learn from it, that if in any equation the quantity $\frac{\log. \sqrt{-1}}{\sqrt{-1}}$ ſhould occur, it may be made to diſappear, by the ſubſtitution of a circular arch, and a property, common to both the circle and hyperbola, may be obtained. The ſame is to be obſerved of the rules which have been invented for the transformation and reduction of impoſſible quantities(e): they facilitate the operations of this imaginary arithmetic, and thereby lead to the knowledge of the moſt beautiful and extenſive analogy which the doctrine of quantity has yet exhibited.

(e) The rules chiefly referred to are thoſe for reducing the impoſſible roots of an equation to the form $A+B\sqrt{-1}$.